

## Perturbative Casimir shifts of dispersive spheres at finite temperature

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2001 J. Phys. A: Math. Gen. 34 5781

(<http://iopscience.iop.org/0305-4470/34/29/304>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.97

The article was downloaded on 02/06/2010 at 09:08

Please note that [terms and conditions apply](#).

# Perturbative Casimir shifts of dispersive spheres at finite temperature

**G Barton**

Centre for Theoretical Physics, University of Sussex, Brighton BN1 9QH, UK

E-mail: g.barton@sussex.ac.uk

Received 25 April 2001

Published 13 July 2001

Online at [stacks.iop.org/JPhysA/34/5781](http://stacks.iop.org/JPhysA/34/5781)

## Abstract

The quantum-electrodynamic Helmholtz free energy of binding  $\Delta B$  is determined to order  $(n\alpha)^2$  for macroscopic spheres of radius  $a$  and dielectric function  $\varepsilon(\omega) = 1 + n\alpha(1 - \omega^2/\Omega^2)^{-1}$ , after renormalization by subtraction of components proportional to volume and to surface area. The method generalizes previous results for  $T = 0$  to realistic temperatures  $kT \ll \hbar\Omega$ , expressing  $\Delta B$  in terms of moments of the standard properly retarded interatomic potential  $W(\rho, \Omega; T)$  at separations  $\rho$ . Divergences are avoided by allowing for a minimum value  $\lambda$  of  $\rho$ , comparable to the radius of the hard core of  $W$ , so that  $\lambda\Omega/c \ll 1$ . The shift  $\Delta B$  is dominated by negative components of order  $-(n\alpha)^2\hbar\Omega \log(c/\Omega\lambda)$ , independent of both  $a$  and  $T$ , such components being generic to the free energy of a single body as opposed to the interaction between bodies that are mutually disjoint. When  $kTa/\hbar c \gg 1$ , the temperature-dependent part of  $\Delta B/(n\alpha)^2$  is of order  $-kT \log(kTa/\hbar c)$ ; when  $kTa/\hbar c \ll 1$ , it is of order  $-(\hbar c/a)(kTa/\hbar c)^3$ .

PACS numbers: 12.20.-m, 02.30.-f, 03.65.-w, 03.70.+k, 11.10.-m

## 1. Introduction

Consider the Helmholtz free energy of binding, call it  $B$ , of a single macroscopic body, more specifically of an optically dilute atomic solid, with  $n$  atoms per unit volume, each atom treated as a simple-harmonic oscillator having dynamical polarizability

$$\alpha(\omega) = \alpha\Omega^2 / (\Omega^2 - (\omega + i0)^2), \quad (1.1)$$

with  $\Omega$  envisaged as a typical atomic excitation frequency. The solid is modelled as a continuum with dielectric function

$$\varepsilon(\omega) \simeq 1 + 4\pi n\alpha(\omega), \quad n\alpha \ll 1, \quad (1.2)$$

and we shall work only to order  $(n\alpha)^2$ . We describe as *nondispersive* the limit  $\Omega \rightarrow \infty$ , where  $\alpha(\omega) \rightarrow \alpha$  and  $\varepsilon(\omega) \rightarrow 1 + 4\pi n\alpha$  become constants independent of the frequency.

For a body with total volume  $V$  and surface area  $S$  it proves convenient to write  $B$  as

$$B = Vu + S\sigma + \Delta B, \quad (1.3)$$

where  $u$  is the free energy of binding per unit volume in unbounded material,  $\sigma$  the surface tension, i.e. the free energy per unit surface area of a halfspace, while we call  $\Delta B$  *the renormalized Casimir free energy*. Here we aim to extend to finite temperatures some earlier conclusions about  $B$  and  $\Delta B$  at  $T = 0$ , especially for a single body as opposed to the interaction between bodies that are mutually disjoint (Barton 1999 and 2001a, cited as I and II; we shall refer also to two papers on the nondispersive limit at  $T > 0$ : Barton 2001b, c, cited as III and IV).

For  $T = 0$ , it is shown in II that  $B$  calculated perturbatively from quantum electrodynamics to order  $(n\alpha)^2$  is identically the same as the result calculated from appropriate moments  $\mathcal{J}_N$  of the properly retarded atom–atom interactions  $W(\rho)$  at separation  $\rho$ , defined so that the interatomic force reads  $-\nabla W$ . We write these as<sup>1</sup>

$$W(\rho) = -\alpha^2 f(\rho), \quad \mathcal{J}_N(\rho) \equiv \int_{\rho}^{\infty} d\rho' f(\rho')\rho'^N. \quad (1.4)$$

In particular, one finds

$$u = -(n\alpha)^2 2\pi \mathcal{J}_2(\lambda), \quad \sigma = (n\alpha)^2 \frac{\pi}{2} \mathcal{J}_3(\lambda); \quad (1.5)$$

and, for a solid sphere of radius  $a$ ,

$$B = -\frac{1}{2}(n\alpha)^2 \int_{r < a} \int_{r' < a} d^3r d^3r' f(\rho) \quad \text{subject to } \rho > \lambda, \quad (1.6)$$

$$\frac{\Delta B}{(n\alpha)^2 \pi^2} = -\frac{1}{6} \Delta \mathcal{J}_5 + \frac{8a^3}{3} \mathcal{J}_2(2a) - 2a^2 \mathcal{J}_3(2a). \quad (1.7)$$

Here

$$\Delta \mathcal{J}_5 \equiv \int_{\lambda}^{2a} d\rho f\rho^5, \quad (1.8)$$

$\lambda$  being a minimum separation<sup>2</sup>, comparable to the hard-core radius of the true interatomic potential, and the only echo in our model of the fact that real solids are atomically granular rather than truly continuous. Notice in (1.7), (1.8) that  $\mathcal{J}_{2,3}(2a)$  depend on the potential only at separations *not* realized inside the sphere, while  $\Delta \mathcal{J}_5$  depends only on separations that *are* realized. Such are the subtleties of renormalization.

The crucial point is that (1.4)–(1.8) apply equally at nonzero temperature, because, in the underlying proof of equivalence<sup>3</sup> with QED, standard thermodynamic perturbation theory prescribes precisely the switch of  $B$  from energy shift at  $T = 0$  to free-energy shift at  $T > 0$ , given by expectation values formed with the zero-order (unperturbed) canonical distribution

<sup>1</sup> We use natural units  $\hbar = 1 = c$  (apart from occasional highlighting), and unrationalized Gaussian units for the Maxwell field (as in III and IV, whereas rationalized units were used in I and II).

<sup>2</sup> It is a truism of condensed-state physics that the interiors of perfectly hard cores do not contribute to the binding.

<sup>3</sup> Appendix E of II writes down the QED Hamiltonian that matches (1.1), (1.2), and proves the equivalence at  $T = 0$ . For nondispersive materials the proof at  $T = 0$  is given in I, and extended to  $T > 0$  in IV. Alternatively, if one considers  $B$  as an object of interest in its own right rather than as a challenge to field theory, then one could reason that such proofs are redundant once the intermolecular forces have been identified in the form  $-\nabla W$ : for then it is self-evident that equation (1.6) represents the free energy of binding, and the other equations ensue automatically. The direct connections between  $B$  and the forces are explicated by Boyer (1975 and 1973).

(see e.g. Landau and Lifshitz 1958, Peierls 1979). Of course one must then use the appropriate  $T$ -dependent potential:

$$f = f(\rho, \Omega; T) \quad \Rightarrow \quad \Delta B = \Delta B(a, \Omega; T); \quad (1.9)$$

however, when it is possible without confusion, we may suppress some or all of the arguments, lapsing e.g. to  $f(\rho, \Omega; T) = f(\rho; T) = f(\rho)$ . Because  $B$  is a Helmholtz free energy, the corresponding entropy  $S$  and total energy  $E$  are given by

$$S = -\partial B/\partial T, \quad E = B + TS, \quad (1.10)$$

and similarly for  $\Delta S$  and  $\Delta E$ .

The present paper determines  $\Delta B(a, \Omega; T)$  for a macroscopic sphere<sup>4</sup> under the evident restriction

$$\lambda \ll 1/\Omega \ll a, \quad (1.11)$$

and at realistic temperatures such that

$$kT \ll \Omega; \quad (1.12)$$

otherwise any real atoms would be almost wholly ionized, and any initially condensed body would have evaporated long since. Part of the challenge is to reduce to some sort of order the complications from having two dimensionless parameter free of the cutoff, say  $kTa/\hbar c$  and  $a\Omega/c$ , instead of just  $a\Omega/c$  as at  $T = 0$ .

It proves convenient to define

$$A \equiv 4\pi kTa/\hbar c, \quad \Lambda \equiv 4\pi kT\lambda/\hbar c, \quad \Lambda/A = \lambda/a \ll 1. \quad (1.13)$$

Moreover, whether  $A$  is large or small, we always have

$$\Lambda \ll 1; \quad (1.14)$$

physically, this follows from  $\hbar\Omega \ll \hbar c/\lambda$  combined with  $kT \ll \hbar\Omega$ . (Formally, (1.14) merely reflects the approach to  $\lambda = 0$ , which is understood throughout.) We also need to define some terminology: in free energies, terms that would diverge in the limit  $\lambda \rightarrow 0$  are called (*nominally*) *divergent*, with the qualification usually omitted; terms that remain finite are called *convergent*; and terms that vanish with  $\lambda$  (or equivalently with  $\Lambda$ ) will always be dropped. *Pure Casimir terms are defined as those convergent terms that depend only on  $n\alpha$ , the radius  $a$ , and  $kT$  (but neither on  $\lambda$  nor on  $\Omega$ ).*

We shall derive analytic approximations for the two extreme regimes compatible with (1.11) and (1.12), namely the regimes where

$$\text{either} \quad \lambda \ll 1/\Omega \ll a \ll 1/kT \Rightarrow A \ll 1: \quad (\text{low } T), \quad (1.15)$$

$$\text{or} \quad \lambda \ll 1/\Omega \ll 1/kT \ll a \Rightarrow A \gg 1: \quad (\text{high } T). \quad (1.16)$$

From such approximations we shall systematically drop all contributions to  $\Delta B$  that are exponentially small in the sense of being at most of order  $\exp(-\hbar\Omega/kT)$  or, when  $A$  is large, of order  $\exp(-A)$ . This simplifies the displays enormously: intermediate values of  $A$  would require extensive numerical calculations. For orientation, equations (1.11)–(1.16) might be viewed in the light of the parameters for solid argon (cf II, appendix F); there, expressed in atomic units, one finds the lengths  $\lambda \simeq 4.4$  and  $c/\Omega \simeq 3.1 \times 10^2$ , while  $1/kT \simeq 1.4 \times 10^5 \times (300/T)$  and  $(a/\text{mm}) \simeq 1.9 \times 10^7$ .

<sup>4</sup> There is interesting physics also in the temperature-dependence of  $u$  and of  $\sigma$ , but renormalization makes it irrelevant to  $\Delta B$ .

The calculations for  $T = 0$  are reported in II. Subject to (1.11), equations (1.7), (1.8) then yield

$$\frac{\Delta B(T = 0)}{(n\alpha)^2\pi^2} = -\frac{1}{6}\Delta\mathcal{J}_5(T = 0) - \frac{23}{4\pi a} \cdot \frac{1}{24}, \quad (1.17)$$

where

$$\Delta\mathcal{J}_5(T = 0) = \mathcal{J}_5(\lambda; 0) - \mathcal{J}_5(2a; 0) = \mathcal{J}_5(\lambda; 0) - \frac{23}{4\pi a} \cdot \frac{1}{2}, \quad (1.18)$$

so that

$$\frac{\Delta B(T = 0)}{(n\alpha)^2\pi^2} = -\frac{1}{6}\mathcal{J}_5(\lambda; 0) + \frac{23}{96\pi a}. \quad (1.19)$$

The divergent component reads

$$\mathcal{J}_5(\lambda; 0) = \Omega \left\{ \frac{3}{4} \left[ \log \left( \frac{1}{2\Omega\lambda} \right) - \gamma \right] + \frac{65}{32} + \mathcal{O} \left( \Omega\lambda \log \left( \frac{1}{\Omega\lambda} \right) \right) \right\}. \quad (1.20)$$

Remarkably, subject to (1.11) this is independent of  $a$ ; which proves important, because we shall find that  $-\mathcal{J}_5(\lambda; 0)/6$  persists unmodified as the only nominally divergent and easily the dominant component<sup>5</sup> of  $\Delta B$  at any temperature subject to (1.12). By contrast, the component  $23/96\pi a$  on the right of (1.19), first derived from QED in I, is a pure Casimir term, and smaller by some orders of magnitude.

Finally, it may be worth anticipating some surprise at just how extensively this study, concentrated on dispersive bodies, can, *a posteriori*, exploit relations derived directly in the far simpler nondispersive limit. For brevity, we shall draw on such results as much as is possible without making the present paper (V) unreadable on its own. Hence we schematize its relation to its antecedents:

	nondispersive	dispersive
$T = 0$	I	II
$T > 0$	$\left\{ \begin{array}{l} \text{potentials :} \\ \Delta B : \end{array} \right.$	$\left\{ \begin{array}{l} \text{III} \\ \text{IV} \end{array} \right.$
		V.

The rest of this paper is laid out as follows. Section 2 specifies the potential  $f(\rho, \Omega; T)$ ; derives the most useful approximations to it, namely (2.8), (2.9) at temperatures low in a sense that needs careful definition; the nondispersive limit  $f_{nd}$ , equation (2.12), which turns out to be appropriate only in the retarded regime; and the asymptotics of  $f_{nd}$ , namely (2.13) for  $A \ll 1$ , and (2.15) for  $A \gg 1$ . (Though at first sight these preliminaries may appear excessive, they prove labour-saving before long.) Section 3 derives our most dramatic conclusions, those for  $A \gg 1$ , with the remarkable logarithmic end-result for  $\Delta B$  given by (3.11), (3.12). Section 4 deals with the relatively small  $T$ -dependent corrections to  $\Delta B$  for  $A \ll 1$ , given by (4.6) as an incipient series in powers of  $A$ . Section 5 contains some brief final comments. The appendix is mainly technical: it exploits the Abel–Plana formula to accelerate the convergence of certain Matsubara series encountered in sections 2 and 3. (Section A.2 is a digression to defuse awkwardness from double poles on an integration contour, which are endemic in potentials between identical atoms, though they happen not to impede the present paper.)

<sup>5</sup> That  $\Delta B$  contains divergent terms has or should have been known ever since the work of Candelas (1982); I and II discuss them in some detail. Inspection will reveal that they dominate  $\Delta B$  not so much because they would diverge (logarithmically) as  $\lambda \rightarrow 0$ , but because they are prefaced by  $\Omega$ . This feature is one of the most characteristic consequences of dispersion (cf II): nondispersively the divergent parts of  $\Delta B$  would have a factor  $1/\lambda$  instead of  $\Omega$ , which would make their dominance even more pronounced (cf IV).

**2. The potential  $f(\rho, \Omega; T)$**

Probably the most accessible of convincing derivations is that given by McLachlan (1963a, b). We adopt the dipole approximation, and quote the particularly convenient (some might say least inconvenient) form of  $f$  as a Matsubara sum, given by Goedecke and Wood (1999; GW in the following). Temporarily restoring  $\hbar$  and  $c$ , their equations (13), (14) expressed in our notation read<sup>6</sup>

$$f = \frac{2kT}{\rho^6} \mathbf{Q}h(x, y), \quad x \equiv 4\pi kT\rho/\hbar c, \tag{2.1}$$

$$y \equiv \frac{2\pi kT}{\hbar\Omega}, \quad w = x/y = 2\Omega\rho/c,$$

$$h(x, y) \equiv \sum' \frac{\exp(-nx)}{(1+n^2y^2)^2}, \quad \sum' \dots \equiv \sum_{n=0}^{\infty} (1 - \frac{1}{2}\delta_{n,0}) \dots; \tag{2.2}$$

$\mathbf{Q}$  is the zero-degree homogeneous differential operator

$$\mathbf{Q} \equiv 3 - 3x\partial + \frac{5}{4}x^2\partial^2 - \frac{1}{4}x^3\partial^3 + \frac{1}{16}x^4\partial^4, \quad \partial \equiv \frac{\partial}{\partial x}, \tag{2.3}$$

with

$$\mathbf{Q} \cdot \frac{1}{x} = \frac{23}{2x}, \quad \mathbf{Q} \cdot 1 = 3, \quad \mathbf{Q} \cdot x = 0 = \mathbf{Q} \cdot x^3, \quad \mathbf{Q} \cdot x^5 = \frac{11}{2}x^5. \tag{2.4}$$

We identify the regimes

$$\text{nonretarded : } w \ll 1; \quad \text{retarded : } w \gg 1. \tag{2.5}$$

Several approximate forms of  $f$  are prominent in applications.

(i) *At exactly zero temperature*,  $f$  reduces to  $f_0$ , given by replacing  $\sum' \rightarrow \int_0^\infty dn$ . Rescaling the integration variable one finds

$$f(\rho; \Omega; T = 0) \equiv f_0 = \frac{\Omega}{\pi\rho^6} \mathbf{Q} \int_0^\infty du \frac{\exp(-uw)}{(1+u^2)^2}; \tag{2.6}$$

$$f_0(w \ll 1) \simeq f_{vdw} \equiv \frac{3\Omega}{4\rho^6}, \quad f_0(w \gg 1) \simeq f_{CP} \equiv \frac{23}{4\pi\rho^7}. \tag{2.7}$$

The functions  $f_{vdw}$  and  $f_{CP}$  are just the leading terms of asymptotic expansions detailed in II, which also expresses  $f_0$  in terms of sine and cosine integrals<sup>7</sup>, and calculates its moments.

(ii) Appendix A.1 derives a decomposition

$$f = f_0 + f_1, \tag{2.8}$$

which is generally valid, but is designed *for temperatures low in the sense that both  $x$  and  $y$  are small*, when  $f_1$  is just a small correction to  $f_0$ . Under those conditions<sup>8</sup>

$$f_1 \simeq \frac{(kT)^6}{\rho} (2\pi)^5 \left\{ \frac{11}{945} + \frac{(-23x^2 + 308y^2)}{12\,600} + \mathcal{O}[x^4, x^2y^2, y^4] \right\}, \tag{2.9}$$

( $x, y \ll 1, w$  arbitrary).

<sup>6</sup> The function  $h$  should not be confused with Planck's constant. GW's  $f(x, y)$  and  $D_x$  are our  $h(x, y)$  and  $\mathbf{Q}$ .  
<sup>7</sup> It is left as an entertainment for the reader to verify that (2.6) here is equivalent to the *prima facie* somewhat different equations (2.1)–(2.3) in II.  
<sup>8</sup> For the extreme nonretarded limit  $c \rightarrow \infty$ , Boyer (1975) and Milonni and Smith (1996; equation (29)) give  $f_1 \simeq f_0 \times 2 \exp(-\hbar\Omega/kT)$ ; in other words this limit admits no corrections small only proportionately to powers of  $kT/\hbar\Omega$ . Ninham and Daicic (1998; equation (22)) propose a leading term proportional like ours to  $(kT/\hbar c)^6/\rho$ , but with a coefficient different in magnitude and sign. However, their expansion of  $f_0$  in the same equation is wrong, for reasons not identifiable from the context or through the authors, and it seems possible that related errors have infected their  $f_1$  as well. The expansion of  $f_0$  for  $\rho\Omega/c \ll 1$  is given e.g. in II, equation (2.4).

(iii) *The nondispersive (nd) limit entails  $y = 0$ ; it yields*

$$h_{nd} \equiv h(x, y = 0) = \sum' \exp(-nx) = \frac{1}{2} \coth\left(\frac{x}{2}\right), \quad (2.10)$$

and

$$f(\rho, \Omega = \infty; T) \equiv f_{nd}(\rho; T) = \frac{2kT}{\rho^6} \mathbf{Q} h_{nd}, \quad (2.11)$$

or in other words<sup>9</sup>

$$f_{nd} = \frac{kT}{\rho^6} \mathbf{Q} \coth(2\pi kT\rho), \quad (2.12)$$

as derived in III with  $\Omega \rightarrow \infty$  implemented at the outset. The standard expansions of  $\coth(z)$  yield the asymptotics:

$$f_{nd}(2\pi kT\rho \ll 1) = \frac{23}{4\pi\rho^7} + \frac{11(2\pi)^5}{945} \cdot \frac{(kT)^6}{\rho} + \mathcal{O}[\rho(kT)^8], \quad (2.13)$$

$$f_{nd}(2\pi kT\rho \gg 1) = \frac{kT}{\rho^6} \{3 + \mathcal{O}[(kT\rho)^4 \exp(-4\pi kT\rho)]\}. \quad (2.14)$$

Evidently the nd limit entails  $w \gg 1$ , so that  $f_{nd}$  applies in (and only in) the retarded regime. Its low- $T$  version (2.13) will prove useful in section 4. Conversely, in its high- $T$  version (2.14) one can, to exponential accuracy, set

$$f_{nd}(2\pi kT\rho \gg 1) \simeq f_{pvdw} \equiv \frac{3kT}{\rho^6}. \quad (2.15)$$

We call  $f_{pvdw}$  the *pseudo-Van-der-Waals potential*. (One might be tempted to call it the ‘classical limit’ since it is free of  $\hbar$ , but this could mislead, because the underlying condition  $y = kT/\hbar\Omega \rightarrow 0$  fails if the limit  $\hbar \rightarrow 0$  is taken before  $\Omega \rightarrow \infty$ .)

For  $y$  small but not strictly zero, and arbitrary  $x$ , the potential is determined in appendix A.1. When  $x$  is small too,  $h$  and  $f$  are already known from (2.8) and (2.9). When  $x$  is large, one has  $f \simeq f_{pvdw}$ , which is merely a special case of the general result stated next.

(iv) Equations (2.1)–(2.4) show that to leading order

$$f \rightarrow f_{pvdw} \quad \text{as } x \rightarrow \infty, \quad \text{or } y \rightarrow \infty, \quad \text{or both}, \quad (2.16)$$

because either limit suffices to reduce  $h(x, y)$  to its first term (with  $n = 0$ ), entailing  $h \rightarrow 1/2$  and  $\mathbf{Q}h \rightarrow 3/2$ . Indeed much of the fascination of finite-temperature Casimir problems stems precisely from the fact that for any fixed  $T$ , however small, large enough  $\rho$  inevitably entails  $f \simeq f_{pvdw}$ , which may be weak in absolute terms, but has become dominant rather than a relatively small correction.

The remarkable numerical work of GW illustrates very clearly how, as  $\rho$  rises,  $f(\rho, \Omega; T)$  modulates from  $f_{vdw}$  to  $f_{CP}$  and then from  $f_{CP}$  to  $f_{pvdw}$ , provided  $kT/\Omega \lesssim 10^{-3}$ ; and how the intermediate plateau with  $f \simeq f_{CP}$  disappears if this ratio though small is not small enough. If such a plateau does exist, then, to a first approximation, the inner transition ( $f_{vdw}$  to  $f_{CP}$ ) is described by  $f(\rho, \Omega; 0)$ , and the outer transition ( $f_{CP}$  to  $f_{pvdw}$ ) by  $f(\rho, \infty; T) = f_{nd}$ .

We are now in a position to tackle the moments needed in (1.7), namely  $\mathcal{J}_{2,3}(2a; T)$  and  $\Delta\mathcal{J}_5(T)$ . Notice from (2.15), (2.16) that, unless  $T$  is exactly zero, one cannot write  $\Delta\mathcal{J}_5(T) = \mathcal{J}_5(\lambda; T) - \mathcal{J}_5(2a; T)$ , because the separate integrals would then diverge at their upper limit.

<sup>9</sup> In III and IV the symbol  $f$  is used for what here we denote as  $f_{nd}$ .

It may be worth stressing that (2.1), (2.2) apply to systems in overall thermal equilibrium; so therefore do the resultant  $B$  and  $\Delta B$ . In particular, the atoms are assumed to be Boltzmann-distributed over their energy eigenstates; hence strictly speaking the polarizability  $\alpha(\omega)$  is a canonical average, and therefore a function of the temperature, introducing a  $T$ -dependence additional to that which we study explicitly. As far as the writer knows, this point of principle was last stressed by McLachlan (1963b). Fortunately we can disregard it, for two quite different reasons. First, in virtue of the restriction (1.12), Boltzmann factors  $\exp(-\Omega/kT)$  ensure that the proportion of excited to ground-state atoms is negligible. Second, for the simple-harmonic oscillators that we consider, the polarizability happens to be the same in all states, so that (1.1) applies with exactly the same  $\alpha$  at any temperature.

Finally, for completeness and without enthusiasm, we quote the potential in a form that is sometimes regarded as more basic than (2.1)–(2.4), and is obtainable from them by rotating an integration contour back from the imaginary to the real axis in the complex frequency plane: namely

$$f(\rho) = -\frac{2kT}{\rho^6} Q \text{Im} \int_0^\infty du \coth(\pi u) \frac{\exp(-iux)}{[1 - y^2 u^2 - i0]^2}. \tag{2.17}$$

Some of the disadvantages attending this version, on account partly of the double pole in its integrand, may become more apparent from the appendix, and especially from section A.2.

### 3. The renormalized free energy $\Delta B(a, \Omega; T)$ at high temperatures: $1/kT \ll 2a$

We must evaluate (1.7) for  $A \gg 1$ . For  $\rho > 2a$ , equation (2.16) then entails  $f \simeq f_{pVdW}$ , whence the moments  $\mathcal{J}_N(2a)$  are given by

$$\mathcal{J}_N \simeq \mathcal{J}_N(\rho; T)_{pVdW} = \left( \frac{3kT}{5 - N} \right) \frac{1}{\rho^{5-N}}. \tag{3.1}$$

Hence

$$\left\{ \frac{8a^3}{3} \mathcal{J}_2(2a; T) - 2a^2 \mathcal{J}_3(2a; T) \right\}_{pVdW} = -\frac{5kT}{12}, \tag{3.2}$$

as known already from IV. This contribution, which at  $T = 0$  is proportional to  $\hbar c/a$ , has become proportional to  $kT$  instead, and independent of  $a$ , because for  $\rho > 2a$  the potential  $f$  has become proportional to  $kT/\rho^6$  instead of  $\hbar c/\rho^7$ .

More effort is needed to determine  $\Delta \mathcal{J}_5$ , because it is sensitive to the potential for all  $\rho$  from  $\lambda$  to  $2a$ . We change the integration variable from  $\rho$  to  $x$ ; notice from (1.13) and (1.16) that in this regime  $\Lambda \ll 1 \ll A$ ; separate the contributions to  $h(x, y)$  from the terms with  $n = 0$  and with  $n \geq 1$ ; and start from

$$\Delta \mathcal{J}_5 = \Delta \mathcal{J}_5(n = 0) + \Delta \mathcal{J}_5(n \geq 1), \tag{3.3}$$

$$\Delta \mathcal{J}_5(n = 0) = 3kT \int_\Lambda^{2A} \frac{dx}{x} = 3kT \log \left( \frac{2a}{\lambda} \right), \tag{3.4}$$

$$\begin{aligned} \Delta \mathcal{J}_5(n \geq 1) &= 2kT \sum_{n=1}^\infty \frac{1}{(1 + n^2 y^2)^2} \\ &\times \int_\Lambda^{2A} \frac{dx}{x} \left[ 3 + 3nx + \frac{5}{4} n^2 x^2 + \frac{1}{4} n^3 x^3 + \frac{1}{16} n^4 x^4 \right] \exp(-nx). \end{aligned} \tag{3.5}$$



In (3.5), the upper limit may be replaced by  $\infty$ , since the differences are at most of order  $\exp(-2A)$ , ie exponentially smaller than the powers that we shall encounter presently. Moreover, we may integrate all but the first term within the square brackets down to  $x = 0$ , since the difference vanishes with  $\Lambda$ , and obtain

$$\int_0^\infty \frac{dx}{x} \left[ 3nx + \frac{5}{4}n^2x^2 + \frac{1}{4}n^3x^3 + \frac{1}{16}n^4x^4 \right] \exp(-nx) = \frac{41}{8}.$$

By contrast, the first term inside the square brackets in (3.5) features the exponential integral

$$\int_\Lambda^\infty \frac{dx}{x} \exp(-nx) = E_1(n\Lambda) \simeq \log\left(\frac{1}{n\Lambda}\right) - \gamma,$$

where  $\gamma \simeq 0.577$  is Euler's constant. These approximations yield

$$\Delta\mathcal{J}_5(n \geq 1) \simeq 2kT \left\{ 3 \log\left(\frac{1}{\Lambda}\right) - 3\gamma + \frac{41}{8} \right\} \sum_{n=1}^\infty \frac{1}{(1+n^2y^2)^2} - 6kT \sum_{n=1}^\infty \frac{\log(n)}{(1+n^2y^2)^2}. \quad (3.6)$$

The first sum is elementary, and easily approximated for small  $y$ :

$$\sum_{n=1}^\infty \frac{1}{(1+n^2y^2)^2} = \frac{\pi}{4y} \coth\left(\frac{\pi}{y}\right) + \frac{\pi^2}{4y^2} \cdot \frac{1}{\sinh^2(\pi/y)} - \frac{1}{2} = \frac{\pi}{4y} - \frac{1}{2} + \mathcal{O}\left(\frac{\exp(-2\pi/y)}{y}\right). \quad (3.7)$$

The second sum in (3.6) is more demanding, and is relegated to appendix A.3; the result reads

$$\Sigma \equiv \sum_{n=1}^\infty \frac{\log(n)}{(1+n^2y^2)^2} \simeq \frac{\pi}{4y} \log\left(\frac{1}{y}\right) - \frac{\pi}{4y} + \frac{1}{2} \log(2\pi), \quad (3.8)$$

up to terms that vanish with  $y$ . From (3.6)–(3.8) one finds

$$\Delta\mathcal{J}_5(n \geq 1) = \frac{3}{4}\Omega \left[ \log\left(\frac{1}{2\Omega\lambda}\right) - \gamma \right] + \frac{65}{32}\Omega - 3kT \left[ \log\left(\frac{1}{2kT\lambda}\right) - \gamma \right] - \frac{41}{8}kT. \quad (3.9)$$

Finally, substitution into (3.3) from (3.4) and (3.9) yields<sup>10</sup>

$$\Delta\mathcal{J}_5(T) = \mathcal{J}_5(\lambda; T=0) + 3kT \left[ \log(4kTa) + \gamma \right] - \frac{41}{8}kT, \quad (3.10)$$

which we have chosen to express using the combination  $\mathcal{J}_5(\lambda; T=0)$  already quoted in (1.20).

From (1.7), (3.2), and (3.10) one can now assemble our end-result for the sphere at high  $T$ : re-instating  $\hbar$  and  $c$  one finds

$$\frac{\Delta B(a, \Omega; T)}{(n\alpha)^2\pi^2} \simeq -\frac{1}{6}\mathcal{J}_5(\lambda; T=0) - \frac{1}{2}kT \left[ \log\left(\frac{4kTa}{\hbar c}\right) + \gamma \right] + \frac{7}{16}kT \quad (3.11)$$

$$= \frac{\Delta B(a, \Omega; 0)}{(n\alpha)^2\pi^2} - \frac{23}{96\pi a} - \frac{1}{2}kT \left[ \log\left(\frac{4kTa}{\hbar c}\right) + \gamma \right] + \frac{7}{16}kT. \quad (3.12)$$

It hardly needs stressing that the way this varies with  $a$  is utterly different from the variation at zero temperature as described by (1.19). Some more detailed comments are also in order.

<sup>10</sup> Though prima facie surprising, the sign of  $\gamma$  in the combination  $[\log(4kTa) + \gamma]$  is not a misprint: it arises from the merger of the familiar combination  $[\log(1/2kT\lambda) - \gamma]$  from (3.9) with the  $\log(2a/\lambda)$  from (3.4).

(i) The dispersion parameter  $\Omega$  enters only through the first term on the right-hand side of (3.11); this term is independent of  $T$  as well as of  $a$ , but dominates the others by a factor of order  $\Omega/kT$ , which is large by assumption. Traditionally this term is disregarded altogether, perhaps in the belief that, being nominally divergent (on account of its factor  $\log(1/2\Omega\lambda)$ ), it ought to be absorbed by some renormalization. Central to the message of II was the argument that this belief is mistaken, in the sense that it is inappropriate to the physics that determines measured binding energies.

(ii) The logarithm, i.e. the dependence on the radius, stems wholly from  $\Delta\mathcal{J}_5$ .

(iii) As we have stressed,  $\Delta B$  is a free-energy shift. From (1.10), the corresponding shifts  $\Delta S$  in entropy and  $\Delta E$  in total energy are given by

$$\begin{aligned} \frac{\Delta S/k}{(n\alpha)^2 \pi^2} &= \frac{1}{2} \left[ \log \left( \frac{4kTa}{\hbar c} \right) + \gamma \right] + \frac{1}{16}, \\ \frac{\Delta E}{(n\alpha)^2 \pi^2} &= -\frac{1}{6} \mathcal{J}_5(\lambda; T=0) + \frac{1}{2} kT. \end{aligned} \tag{3.13}$$

#### 4. The renormalized free energy $\Delta B(a, \Omega; T)$ at low temperatures: $1/kT \gg 2a$

We must evaluate (1.7) for  $A \ll 1$  and  $y \ll 1$ . From section 2 paragraph (iii) we see that  $y \ll 1$  admits  $f \simeq f_{nd}$ ; but now, instead of calculating the  $\mathcal{J}_{2,3}(2a)$  with the asymptotic form  $f_{pvdw}$  of  $f_{nd}$  (as in section 3), we must use in its exact form (2.12), which makes the integrations quite awkward. They are reported in IV, whence we merely quote the results:

$$\mathcal{J}_2(2a) = \frac{1}{\pi a^4} \left\{ \frac{23}{256} + \frac{7A^4}{11\,520} - \frac{11A^6}{60\,480} + \mathcal{O}(A^8) \right\}, \tag{4.1}$$

$$\mathcal{J}_3(2a) = \frac{1}{\pi a^3} \left\{ \frac{23}{96} + \frac{\zeta(3)A^3}{32\pi^2} - \frac{11A^6}{45\,360} + \mathcal{O}(A^8) \right\}. \tag{4.2}$$

The calculation of  $\Delta\mathcal{J}_5$  runs quite differently, because it is sensitive to values of  $\rho$  all the way down to  $\lambda$ . The trick is to split  $f = f_0 + f_1$  as for low  $T$  in (2.8), (2.9), and correspondingly split<sup>11</sup>

$$\Delta\mathcal{J}_5 = \Delta\mathcal{J}_5(T=0) + \Delta\mathcal{J}_{51} \tag{4.3}$$

into a zero-temperature part from (1.18) plus the  $T$ -dependent correction

$$\Delta\mathcal{J}_{51} = \int_{\lambda}^{2a} d\rho \rho^5 f_1 \simeq \frac{11(2\pi)^5}{945} (kT)^6 \int_0^{2a} d\rho \rho^4 = \frac{11}{18\,900\pi a} A^6. \tag{4.4}$$

Once again we have set  $\lambda \rightarrow 0$ , because the difference vanishes with  $\lambda$ .

We can now substitute into (1.7) from (4.1)–(4.4), and find, with  $\Delta B(T=0)$  from (1.17),

$$\Delta B = \Delta B(T=0) + \Delta B_1, \tag{4.5}$$

$$\frac{\Delta B_1}{(n\alpha)^2 \pi^2} \simeq \frac{1}{a} \left\{ -A^3 \frac{\zeta(3)}{16\pi^3} + A^4 \frac{7}{4320\pi} - A^6 \frac{11}{113\,400\pi} + \mathcal{O}(A^8) \right\}. \tag{4.6}$$

The first term on the right stems from  $\mathcal{J}_3$ , the second from  $\mathcal{J}_2$ , and the third jointly from  $\mathcal{J}_2$ ,  $\mathcal{J}_3$ , and  $\Delta\mathcal{J}_{51}$ .

<sup>11</sup> Do not confuse  $\Delta\mathcal{J}_5(T=0)$  and  $\Delta\mathcal{J}_{51}$  with  $\Delta\mathcal{J}_5(n=0)$  and  $\Delta\mathcal{J}_5(n \geq 1)$  in (3.3)–(3.5).

## 5. Comments

(i) We have extended to nonzero temperatures the perturbative methods for the Casimir energy of a *single, macroscopic, and dispersive dielectric body*, as developed in II and as applied there to several shapes, but only at  $T = 0$ . From another point of view, we have extended the finite-temperature theory in IV from nondispersive to dispersive materials. The restriction to insulators is essential: for instance, our method is quite unwarranted for a plasma with dielectric function  $\varepsilon = 1 - \Omega^2/\omega^2$ , which cannot be treated as a set of *localized* oscillators, no matter how tenuous it might be.

(ii) The main surprise for the writer is just how narrowly the conditions  $kT \ll \hbar\Omega$  and  $a\Omega/c \gg 1$  restrict the appearances of the dispersion parameter  $\Omega$  in the end-results (3.11) and (4.5), (4.6) for the shift  $\Delta B$ . Much of this simplicity ensues from dropping all contributions that section 1 has called exponentially small. When  $kTa/\hbar c$  is very large, and again when it is very small,  $\Delta B$  splits naturally into two terms. In each case the second term is a small correction, independent of  $\Omega$  and  $\lambda$ , and it catches the eye. But the first term is larger by several orders of magnitude: it is nominally divergent, a function of  $\Omega$  and of  $\lambda$  but not of  $a$ , and already familiar from II. In (4.5) it is just  $\Delta B(T = 0)$ , while in (3.11), (3.12) it differs from  $\Delta B(T = 0)$  merely by the convergent term  $-(n\alpha)^2 23\pi/96a$ .

(iii) For effects that depend jointly on  $\Omega$  and on low  $T$ , one would have to look to contributions from the *second* term in the correction (2.9) to the potential.

(iv) As already stressed in section 3 comment (i), and in comment (ii) just above,  $\Delta B$  is dominated by its nominally divergent zero-temperature component. This is negative, and in that sense attractive; and so consequently is  $\Delta B$  itself, at any temperature. As discussed in II, it is far from clear what if anything our expressions for  $\Delta B$  might imply about local stresses in the sphere; nevertheless it may be worth pointing out that the sign of  $\partial\Delta B/\partial a$  never changes either. At low  $T$ , equations (4.5), (4.6) show that the dominant term to depend on  $a$  comes from  $\Delta B(T = 0)$ , whence  $\partial\Delta B/\partial a \simeq -\pi^2(n\alpha)^2 23/96\pi a^2$ , a conclusion unaffected by the possibly more glamorous  $T$ -dependent correction  $\Delta B_1$ ; while at high  $T$ , equation (3.11) yields  $\partial\Delta B/\partial a \simeq -kT/2a$ .

(v) Revert now to point (ii). Though essential to the physics, dispersion enters the end-results in ways so restricted that one is strongly motivated to explore the nondispersive model in its own right, taking  $\alpha$  as independent of frequency from the start. The temptation is the greater because with the potential  $f_{nd}$ , unrealistic though it might be, all calculations can at least be done in closed form; moreover,  $f_{nd}$  itself can then be derived from QED with relative ease, and without Matsubara expansions. Finally, to whatever extent one might be justified in ignoring dispersion, it becomes possible to deal in much the same way with magnetic as we have here dealt with electric polarizability. (In particular, one can then deal with materials having  $\varepsilon\mu = 1$ , which are of some interest to theorists.) The resulting  $T$ -dependent potentials are reported in III, and the consequent shifts  $\Delta B$  in IV.

(vi) The writer knows of no other attempts to derive the finite- $T$  results reported here for dispersive media. (Høye *et al* (2000) tackle the problem of two concentric spheres separated by a narrow gap; but they switch from weak to perfect reflectors before deriving anything explicit.) The nearest possibility is to compare the *nondispersive* results (from IV) with the output of mode-summation methods reported by Nesterenko *et al* (2001) and by Klich *et al* (2000), for media having  $\mu = 1$  and  $\mu = 1/\varepsilon$  respectively. These comparisons are spelled out in IV, together with the reasons why it remains unclear, for the present, whether one should expect exact coincidence, and what physics would be implied by any discrepancies.

**Acknowledgments**

It is a pleasure to acknowledge the advice of Iver Brevik, Claudia Eberlein, and Peter Milonni.

**Appendix. Applications of the Abel–Plana formula**

We quote the formula restricted to summands  $F(n)$  free of singularities for  $\text{Re } n > m$ , where it reads

$$\sum_{n=m}^{\infty} (1 - \frac{1}{2}\delta_{nm})F(n) - \int_m^{\infty} dn F(n) = i \int_0^{\infty} \frac{du}{[\exp(2\pi u) - 1]} [F(m + iu + \epsilon) - F(m - iu + \epsilon)]. \tag{A.1}$$

The infinitesimal positive addends in  $\pm iu + \epsilon = \pm i(u \mp i\epsilon)$  stem from the contour rotations that led to the formula in the first place. They prescribe how to avoid any poles on the vertical  $n = m \pm iu$  through  $m$  in the complex  $n$  plane.

The formula is perhaps more often exploited in problems where the sum and the integral on the left diverge, and is discussed in that connection elsewhere (Barton 1981, 1982). *A fortiori* it holds when both converge, continuing in our case to serve its primary purpose by effectively accelerating the convergence of the series for  $h$ , as will appear presently. For the basic mathematics see Hardy (1949).

*A.1. The potential  $f$  at low temperature*

For use in (2.2) the formula (with  $m = 0$ ) gives the exact relations<sup>12</sup>

$$h = h_0 + h_1, \quad h_0 \equiv \int_0^{\infty} dn \frac{\exp(-nx)}{(1 + y^2n^2)^2} = \frac{1}{y} \int_0^{\infty} du \frac{\exp(-wu)}{(1 + u^2)^2}, \tag{A.2}$$

$$h_1 = i \int_0^{\infty} \frac{du}{[\exp(2\pi u) - 1]} \left\{ \frac{\exp(-iux)}{[1 - y^2u^2 + i\epsilon]^2} - \frac{\exp(iux)}{[1 - y^2u^2 - i\epsilon]^2} \right\}. \tag{A.3}$$

The term  $h_0$  is responsible for the familiar zero-temperature  $f_0$  in the decomposition (2.8). This calculation is standard, and we do not discuss it further, being concerned only with the correction term  $h_1$ , which is responsible for  $f_1$ .

At first sight, the double poles infinitesimally close to the integration path threaten trouble, but the next section shows (a) how to make them amenable to calculation in case of need (though here there will be no need); and (b) that for  $y \ll 1$ , (as here), their vicinity contributes only terms with factors  $\exp(-2\pi/y)$ , i.e. of an order that we are disregarding in any case. To this accuracy the factors  $[1 - y^2u^2 \pm i\epsilon]^{-2}$  become effectively identical; we expand them in powers of  $y^2u^2$ ; purely for book-keeping purposes replace  $u^{2n}$  inside by  $(-d^2/dx^2)^n$  outside the integral; and re-sum to obtain the symbolic but very useful expression

$$h_1 \simeq \frac{2}{[1 + y^2d^2/dx^2]^2} \int_0^{\infty} \frac{du \sin(ux)}{[\exp(2\pi u) - 1]} = \frac{1}{[1 + y^2d^2/dx^2]^2} \left\{ \frac{1}{2} \coth\left(\frac{x}{2}\right) - \frac{1}{x} \right\}. \tag{A.4}$$

Expanding  $\{ \dots \}$  for large  $x$  one has

$$h_1 \simeq \frac{1}{[1 + y^2d^2/dx^2]^2} \left\{ \frac{1}{2} - \frac{1}{x} + \mathcal{O}(e^{-x}) \right\};$$

<sup>12</sup> In this appendix the subscript 0 identifies  $T = 0$ ; it should not be confused with the term having  $n = 0$  in the Matsubara sum (2.2).

with the prefactor re-expanded this eventually reproduces the expected result  $f_1 \simeq f_{pvdw} - f_{CP}$ , entailing  $f \simeq f_{pvdw}$ .

Here however we are concerned with small  $x$ . Then we can expand either  $\{ \dots \}$  on the right of (A.4), or equivalently  $\sin(ux)$  already under  $\int du \dots$ . Expanding simultaneously in powers of  $y^2$  as well, and acting with  $\mathbf{Q}$ , one obtains

$$\mathbf{Q}h_1 = \frac{11x^5}{2^6 \cdot 945} + \frac{x^5}{2^9 \cdot 1575} [-23x^2 + 308y^2] + \dots, \quad (\text{A.5})$$

where the terms proportional to  $x$  and to  $x^3$  have cancelled in virtue of (2.4). Equations (2.1)–(2.4) then yield the expansion quoted in (2.9). It bears repeating that we have assumed  $T$  to be low only in the sense that both  $x$  and  $y$  are small, while  $x/y$  remains arbitrary; hence (2.9) applies equally in the retarded and in the nonretarded regimes.

### A.2. Exorcising the double poles

It may not be immediately obvious how, in practice, to carry the integrations in (A.3) past the double poles infinitesimally close to  $u = 1/y$ . To put the integrals into manageable form we write

$$\eta \equiv 1/y, \quad h_1 = -\eta^4 \frac{\partial H_1}{\partial \eta^2}, \quad (\text{A.6})$$

$$\begin{aligned} H_1 &= 2 \operatorname{Re} i \int_0^\infty \frac{du}{[\exp(2\pi u) - 1]} \cdot \frac{\exp(-iux)}{[\eta^2 - u^2 + i\epsilon]} \\ &= 2 \operatorname{Re} i \int_0^\infty \frac{du \exp(-iux)}{[\exp(2\pi u) - 1]} \left\{ \frac{\mathcal{P}}{[\eta^2 - u^2 + i\epsilon]} - i\pi \delta[u^2 - \eta^2] \right\}, \\ H_1 &= y \frac{\cos(x/y)}{[\exp(2\pi/y) - 1]} + 2\mathcal{P} \int_0^\infty \frac{du}{[\exp(2\pi u) - 1]} \cdot \frac{\sin(ux)}{[\eta^2 - u^2]}. \end{aligned} \quad (\text{A.7})$$

This solves the problem: we drop the first term because it is exponentially small, and the principal-value integral is a differentiable function of  $x$  and  $y$ . In particular, for large  $\eta$  (small  $y$ ) it may be expanded in powers of  $y^2$ , just as in section A.1

### A.3. The sum $\Sigma$

We apply (A.1) (with  $m = 1$ ) to the sum in (3.8), which is required up to terms that vanish with  $y$ . Thus,

$$\Sigma \equiv \sum_1^\infty \frac{\log(n)}{(1 + n^2 y^2)^2} = \Sigma_0 + \Sigma_1, \quad (\text{A.8})$$

where

$$\Sigma_0 = \int_1^\infty \frac{dn \log(n)}{(1 + n^2 y^2)^2} = \frac{\pi}{4y} \log\left(\frac{1}{y}\right) - \frac{\pi}{4y} + 1 + \dots, \quad (\text{A.9})$$

while

$$\Sigma_1 = i \int_0^\infty \frac{du}{[\exp(2\pi u) - 1]} \left\{ \frac{\log(1 + iu)}{[1 + y^2(1 + iu)^2]^2} - \frac{\log(1 - iu)}{[1 + y^2(1 - iu)^2]^2} \right\}. \quad (\text{A.10})$$

Except that we have already approximated  $\Sigma_0$ , equations (A.8)–(A.10) are exact. The crucial point is that, for small  $y$ , the denominators within the braces in (A.10) supply corrections

which are at most of relative order  $y^2$ ; hence we can replace these denominators by unity, and write

$$\Sigma_1 \simeq i \int_0^\infty \frac{du}{[\exp(2\pi u) - 1]} 2i \tan^{-1}(u) = -1 + \frac{1}{2} \log(2\pi). \quad (\text{A.11})$$

Equation (3.8) follows on substituting from (A.9) and (A.11) into (A.8).

## References

- Barton G 1981 *J. Phys. A: Math. Gen.* **14** 1009  
 —1982 *J. Phys. A: Math. Gen.* **15** 323  
 —1999 *J. Phys. A: Math. Gen.* **32** 525 (cited as I)  
 —2001a *J. Phys. A: Math. Gen.* **34** 4083 (cited as II)  
 —2001b The long-range Casimir–Polder–Feinberg–Sucher intermolecular potential at nonzero temperature *Phys. Rev. A* at press (cited as III)  
 —2001c Perturbative Casimir shifts of nondispersive spheres at finite temperature *Phys. Rev. A* at press (cited as IV)  
 Boyer T H 1973 *Phys. Rev. A* **7** 1832  
 —1975 *Phys. Rev. A* **11** 1650  
 Candelas P 1982 *Ann. Phys., NY* **143** 179  
 Goedecke G H and Wood R C 1999 *Phys. Rev. A* **60** 2577  
 Hardy G H 1949 *Divergent Series* (Oxford: Oxford University Press) chapter 13  
 Høye J S, Brevik I and Aarseth J B 2001 *Phys. Rev. E* **63** 051101  
 (Høye J S, Brevik I and Aarseth J B 2000 *Preprint* arXiv quant-ph/0008088 (v2))  
 Klich I, Feinberg J, Mann A and Revzen M 2000 *Phys. Rev. D* **62** 045017  
 Landau L D and Lifshitz E M 1958 *Statistical Physics* (Oxford: Pergamon) section 32  
 McLachlan A D 1963a *Proc. R. Soc. A* **271** 387  
 —1963b *Proc. R. Soc. A* **274** 80  
 Milonni P W and Smith A 1996 *Phys. Rev. A* **53** 3484  
 Nesterenko V V, Lambiase G and Scarpetta G 2000 *Preprint* arXiv hep-th/0006121 (v2, revised since) (*Phys. Rev. D* at press)  
 Ninham B W and Daicic J 1998 *Phys. Rev. A* **57** 1870  
 Peierls R 1979 *Surprises in Theoretical Physics* (Princeton, NJ: Princeton University Press) section 3.3